An introduction to elliptic corner problems via the example of polygonal metamaterials

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Trending topics in spectral theory

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What it is about

Two blocks in title

1. Polygonal metamaterials
2. Elliptic corner problems

Very short explanation.

1. Polygonal Metamaterials:
   Problems of type $\mathscr{P} = -\operatorname{div} A\, \mathbf{grad}$ with sign-changing coefficient A.
   A piecewise smooth in polygonal subdomains.
2. Elliptic corner problems: the "Standard Model" [SM] consists in:
   Fredholm theorems in corner domains of various types (conical, edge, polyhedra,...) based on a hierarchy of elliptic symbols (interior, boundary, corner Mellin, edges,...)

Difficulty to match 1 and 2:

- The metamaterial problem $\mathscr{P}$ is not coercive, not semi-bounded.
- $\mathscr{P}$ is a transmission problem and its ellipticity properties are not obvious.
- In contrast with coercive problems, it may happen that $\mathscr{P}$ is not Fredholm at the variational level.

Global outline

Outline for current section

Standard model of statement

Let $\mathfrak{Q}$ be an operator associated with a boundary value problem of order 2 in a polyhedral domain Ω in $\mathbb{R}^3$ (or $\mathbb{R}^2$ – polygonal domain).

In a very general imprecise form, one of the main statements takes the form

Theorem 0

The operator $\mathfrak{Q}$ is Fredholm if and only if it is elliptic.

This makes sense if we define

1. The pair of functional spaces between which $\mathfrak{Q}$ is Fredholm: Typically, these pairs involve pairs of Sobolev spaces $\big(H^{s+2}(\Omega)\,,\ H^{s}(\Omega)\big)$, or a huge variety of weighted spaces...
2. The ellipticity is a property that has to be satisfied at each point of $\overline{\Omega}$. It takes different forms
   - Inside the domain (classical)
   - On a regular point of the boundary
   - Inside an edge
   - At a corner

The theory of singularities addresses the default of invertibility or Fredholmness.

A short selection of references to SM: Fundamentals

V. A. KONDRAT'EV
Boundary-value problems for elliptic equations in domains with conical or angular points.
Trans. Moscow Math. Soc. **16** (1967) 227–313.

V. G. MAZ'YA, B. A. PLAMENEVSKII
Elliptic boundary value problems on manifolds with singularities.
Probl. Mat. Anal. **6** (1977) 85–142.

V. A. KOZLOV, V. G. MAZ'YA, J. ROSSMANN
Elliptic boundary value problems in domains with point singularities.
Mathematical Surveys and Monographs, 52. American Mathematical Society, 1997.

V. MAZ'YA AND J. ROSSMANN
Elliptic equations in polyhedral domains.
Mathematical Surveys and Monographs, 162. American Mathematical Society, 2010.

A short selection of references to SM: Ψ-do calculus

S. REMPEL, B. W. SCHULZE
Asymptotics for Elliptic Mixed Boundary Problems.
Akademie-Verlag, 1989.

B. W. SCHULZE
Pseudo-differential operators on manifolds with singularities.
Studies in Mathematics and its Applications, Vol. 24. North-Holland, 1991.

B.-W. SCHULZE
Boundary value problems and singular pseudo-differential operators.
Pure and Applied Mathematics (New York). John Wiley & Sons Ltd., 1998.

R. B. MELROSE
Pseudodifferential operators, corners and singular limits,
Proc. International Congress of Mathematicians, Math. Soc. Japan, (1991), 217–234.

R. B. MELROSE
Calculus of conormal distributions on manifolds with corners,
Internat. Math. Res. Notices (1992), no. 3, p. 51–61.

R. B. MELROSE
Differential analysis on manifolds with corners ,
http://www-math.mit.edu/ rbm/book.html

A short selection of references to SM: Grisvard

P. GRISVARD
Problèmes aux limites dans les polygones. Mode d'emploi.
Bull. Dir. Etud. Rech., Sér. C **1** (1986) 21–59.

P. GRISVARD
Singularités en élasticité.
Arch. Rational Mech. Anal. **107 (2)** (1989) 157–180.

P. GRISVARD
Boundary Value Problems in Non-Smooth Domains.
Pitman, London 1985.

A short selection of references to SM: 'CoDaNi'

M. DAUGE
Elliptic Boundary Value Problems in Corner Domains.
Lecture Notes in Mathematics, Vol. 1341. Springer-Verlag, 1988.

S. NICAISE
Polygonal interface problems.
Methoden und Verfahren der Mathematischen Physik, 39. Verlag Peter D. Lang, 1993.

M. COSTABEL, M. DAUGE
General edge asymptotics of solutions of second order elliptic boundary value problems.
Proc. Royal Soc. Edinburgh **123A** (1993) 109–155 and 157–184.

M. COSTABEL, M. DAUGE
Singularities of electromagnetic fields in polyhedral domains.
Arch. Rational Mech. Anal. **151**(3) (2000) 221–276.

M. COSTABEL, M. DAUGE, S. NICAISE
Singularities of Maxwell interface problems.
M2AN Math. Model. Numer. Anal. **33**(3) (1999) 627–649.

A short selection of references to SM: Analytic regularity

I. BABUŠKA, B. GUO
Regularity of the solution of elliptic problems with piecewise analytic data. I & II,
SIAM J. Math. Anal., 19 (1988) 172–203, & 20 (1989) 763–781

M. COSTABEL, M. DAUGE, AND S. NICAISE
Analytic regularity for linear elliptic systems in polygons and polyhedra,
Math. Models Methods Appl. Sci., **22** (2012), 1250015, 63p.

M. COSTABEL, M. DAUGE, AND S. NICAISE
GLC project (Grand Livre des Coins),
?? (20??)

Outline for current section

Standard div A grad operators

Let Ω be a (polygonal or polyhedral) domain. Let

$$x \longmapsto A(x)$$

be a real $L^\infty(\Omega)$ function given on Ω.
We can define the operator $\mathscr{P} \equiv -\operatorname{div} A\,\mathbf{grad}$ in variational form:

$$\begin{array}{cccc} \mathscr{P}: & H^1(\Omega) & \longrightarrow & (H^1(\Omega))' \\ & u & \longmapsto & \left(v \ni H^1(\Omega) \mapsto \int_\Omega A(x)\,\nabla u(x) \cdot \nabla v(x)\,\mathrm{d}x \right) \end{array}$$

In usual applications A is *positive* on $\overline{\Omega}$, smooth, or piecewise smooth in a polygonal or polyhedral partition (Ω_i) of Ω (transmission problem). In this case, the bilinear form

$$\mathscr{A}(u,v) := \int_\Omega A(x)\,\nabla u(x) \cdot \nabla v(x)\,\mathrm{d}x$$

is coercive on $H^1(\Omega)$:

$$\mathscr{A}(u,u) \geq \beta |u|^2_{H^1(\Omega)}, \quad \text{with} \quad \beta > 0$$

hence $\mathscr{P} + \mathbb{I}$ is invertible, and $\mathscr{P}$ is Fredholm of index 0.

Sign-changing operators

When “metamaterials” are involved, A is piecewise smooth, never 0, but will take the positive or the negative sign in different subdomains.
Then we do not have coercivity any more.
A formalism equivalent to the stability of $\mathscr{P}$ was introduced, see e.g.

A.-S. Bonnet-Ben Dhia, L. Chesnel, P. Ciarlet, Jr
T-coercivity for scalar interface problems between dielectrics and metamaterials,
ESAIM Math. Model. Numer. Anal. 46 (2012) no. 6, 1363-1387.

T-coercivity is the existence of a bounded operator $T : H^1(\Omega) \to H^1(\Omega)$ such that

$$\mathscr{A}(u, Tu) \geq \beta |u|^2_{H^1(\Omega)}, \quad \text{with} \quad \beta > 0.$$

The construction of T has to be done for each separate case. It is done for 2d corners in the above reference, generalizing the right-corner case done in

S. Nicaise, J. Venel
A posteriori error estimates for a finite element approximation of transmission problems with sign changing coefficients,
J. Comput. Appl. Math, 235 (2011) no. 1, 4272-4282.

Our typical example of polygonal metamaterial

We consider the operator $\mathscr{P} \equiv -\operatorname{div} A\,\mathbf{grad}$ on Ω as above, where:

- Ω is smooth (for simplicity), and is the disjoint union of two polygonal subdomains Ω_a and Ω_b:

$$\overline{\Omega} = \overline{\Omega}_a \cup \overline{\Omega}_b \quad \text{and} \quad \Omega_a \cap \Omega_b = \emptyset$$

  Assume for simplicity $\Omega_b \Subset \Omega$. Then the interface Γ coincides with $\partial\Omega_b$.

- A is smooth and positive in Ω_a, smooth and negative in Ω_b:

$$A(x) = \begin{cases} a(x) & \text{if } x \in \Omega_a, \quad a \in \mathscr{C}^\infty(\overline{\Omega}_a) \text{ and } a > 0 \text{ in } \Omega_a \cup \partial\Omega \\ b(x) & \text{if } x \in \Omega_b, \quad b \in \mathscr{C}^\infty(\overline{\Omega}_b) \text{ and } b < 0 \text{ in } \Omega_b \end{cases}$$

By Ω_a / Ω_b polygonal, we mean that at any point x of its boundary, it is locally smoothly diffeomorphic either

1. to a half-plane $\mathbb{R} \times \mathbb{R}_+$ (x a smooth boundary/interface point)
2. to a sector K (K plane sector in $\mathbb{R}^2$, $\neq$ half-plane; x is a corner)

Examples for Ω_b: curvilinear triangles, quadrilaterals... $\Subset$ in a disk Ω, and its complementary in Ω for Ω_a.

Outline for current section

Systems of order 2

We are going to give elements of proof for

Theorem 1

Let $\mathscr{L}$ be given by a $N \times N$ system of PDE operators of order 2 with smooth coefficients on a compact manifold Ω without boundary (of dim. d). Then

$\mathscr{L}$ is Fredholm if and only if $\mathscr{L}$ is elliptic

Meaning of terms in the statement:

1. Fredholm from $\boldsymbol{H}^2(\Omega) := H^2(\Omega)^N$ into $\boldsymbol{L}^2(\Omega) := L^2(\Omega)^N$
2. $\mathscr{L} = (\mathscr{L}_{ij}(x, -i\partial_x))_{1\le i,j\le N}$, $\mathscr{L}_{ij}$ smooth in x, polynomial of deg. 2 in $-i\partial_x$.
   - principal parts $\mathscr{L}_{ij}^{\mathrm{pr}}$ defined by removing the terms of order 1 or 0.
   - (principal) symbol σ defined as

$$\sigma(x,\xi) = \left(\mathscr{L}_{ij}^{\mathrm{pr}}(x,\xi)\right)_{1\le i,j\le N}$$

   - ellipticity: $\forall x \in \Omega,\ \ \forall \xi \in \mathbb{S}^{d-1},\ \ \sigma(x,\xi)$ invertible $N \times N$ matrix.

Example of scalar Laplacian: $\mathscr{L} = -\Delta,\ \ \sigma(x,\xi) = |\xi|^2$.

Elliptic $\Longrightarrow$ Fredholm

Choose $x_0 \in \Omega$. By ellipticity, $\xi \mapsto \sigma(x_0, \xi)$ is homogeneous of degree 2 and invertible for any $\xi \neq 0$ in $\mathbb{R}^d$. Consider the discrete symbol

$$\mathbb{Z}^d \ni p \mapsto \sigma(x_0, p+\tfrac{1}{2}), \quad \text{with} \quad p = (p_1, \ldots, p_d), \quad p+\tfrac{1}{2} = (p_1+\tfrac{1}{2}, \ldots, p_d+\tfrac{1}{2})$$

It is invertible for any $p \in \mathbb{Z}^d$. By discrete Fourier transform we deduce that

$$\widetilde{\mathscr{L}}_{x_0} := \sigma(x_0, -i\partial_y + \tfrac{1}{2}) \quad \text{isomorphism} \quad \boldsymbol{H}^2(\mathbb{T}^d) \to \boldsymbol{L}^2(\mathbb{T}^d)$$

(trick borrowed from [KMR1997]), with $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

Main steps of proof of Elliptic $\Longrightarrow$ Fredholm:

1. $\widetilde{\mathscr{L}}_{x_0}$ has the same principal part as $\mathscr{L}_{x_0}$, the operator $\mathscr{L}$ frozen at x_0, i.e. $\widetilde{\mathscr{L}}^{\text{pr}}_{x_0} = \mathscr{L}^{\text{pr}}_{x_0}$
2. In a small enough ball $B(x_0, \rho_{x_0})$, the operators $\mathscr{L}^{\text{pr}}_{x_0}$ and $\mathscr{L}^{\text{pr}}$ are close in $\mathfrak{L}(\boldsymbol{H}^2, \boldsymbol{L}^2)$ norm.
3. There exists a perturbation $\mathscr{L}^{\sharp}_{x_0}$ of $\widetilde{\mathscr{L}}_{x_0}$, still invertible, that has the same principal part as $\mathscr{L}$ in $B(x_0, \rho_{x_0})$.
4. We make a finite covering of Ω by balls $B(x_0, \rho_{x_0})$, $x_0 \in F$, finite subset of Ω
5. The collection of the inverses $(\mathscr{L}^{\sharp}_{x_0})^{-1}$, $x_0 \in F$, leads to the construction of a parametrix $\mathscr{E}$ (pseudo-inverse) for $\mathscr{L}$, i.e.

$$\mathscr{L}\mathscr{E} = \mathbb{I} + \mathscr{K}, \;\; \mathscr{K} \text{ compact in } \boldsymbol{L}^2(\Omega) \qquad \mathscr{E}\mathscr{L} = \mathbb{I} + \mathscr{K}', \;\; \mathscr{K}' \text{ compact in } \boldsymbol{H}^2(\Omega)$$

Non-Elliptic $\Longrightarrow$ Non-Fredholm

Non-ellipticity means that there exist

$$x_0 \in \Omega, \quad \xi_0 \in \mathbb{R}^d \setminus \{0\}, \quad w_0 \in \mathbb{C}^N \setminus \{0\}, \quad \sigma(x_0, \xi_0)\, w_0 = 0.$$

By homogeneity $\sigma(x_0, \lambda\xi_0) = \lambda^2 \sigma(x_0, \xi_0)$, hence

$$\forall \lambda \in \mathbb{R}, \quad \sigma(x_0, \lambda\xi_0)\, w_0 = 0.$$

Define for $n \geq 1$, with notation $\langle \xi, x \rangle = \xi_1 x_1 + \cdots + \xi_d x_d$:

$$u_n(x) = \chi\left(\sqrt{n}\,|x - x_0|\right)\, e^{in\langle \xi_0, x\rangle}\, w_0$$

for a cut-off function $\chi \in \mathscr{C}_0^\infty(\mathbb{R})$, $\chi \equiv 1$ in a neighborhood of 0. Note that

$$\mathscr{L}^{\mathrm{pr}}(x_0, -i\partial_x)(e^{in\langle \xi_0, x\rangle}\, w_0) = 0, \quad \forall n$$

and $\chi\left(\sqrt{n}\,|x - x_0|\right)$ localizes around x_0 at a slower scale than the frequency.
Then the sequence $(u_n)_{n\geq 1}$ is a Weyl sequence for $\mathscr{L}$:

$$\frac{\|\mathscr{L} u_n\|_{L^2(\Omega)} + \|u_n\|_{H^1(\Omega)}}{\|u_n\|_{H^2(\Omega)}} \to 0 \quad \text{as} \quad n \to \infty$$

contradicting *a priori* estimates that hold for a Fredholm operator.

Theorem 1 is proved. □

Outline for current section

Typical metamaterial example

Come back to our operator

$\mathscr{P} = -\operatorname{div} A\, \mathbf{grad}$ with $A = a > 0$ in $\Omega_a \cup \partial\Omega$ and $A = b < 0$ in Ω_b.

Our objective is to prove:

Theorem 2.1

Under hypotheses of slide 10, assume moreover that the interface Γ is $\mathscr{C}^\infty$. Then $\mathscr{P}$ is Fredholm $H^1(\Omega) \to H^1(\Omega)'$ if and only if

$$x \mapsto a(x), \quad x \mapsto b(x), \quad x \mapsto a(x) + b(x) \quad \text{are never 0 on } \Gamma$$

Here, when $x \in \Gamma$, $a(x) = \lim_{y\to x,\ y\in\Omega_a} a(y)$ and $b(x) = \lim_{y\to x,\ y\in\Omega_b} b(y)$

Pick $u \in H^1(\Omega)$ such that $\mathscr{P} u = F$ with $F(v) = (f, v)$ for $f \in L^2(\Omega)$. Then

$$\begin{cases} -\operatorname{div} a\, \mathbf{grad}\, u = f & \text{in} \quad \Omega_a \\ -\operatorname{div} b\, \mathbf{grad}\, u = f & \text{in} \quad \Omega_b \\ a\, \partial_n u = 0 & \text{on} \quad \partial\Omega \\ a\, \partial_{n_a} u + b\, \partial_{n_b} u = 0 & \text{on} \quad \Gamma \end{cases}$$

with n, n_a, n_b exterior unit normals to Ω, Ω_a, Ω_b, respectively.

Symbols at the interface

Since $a > 0$ in Ω_a and $b < 0$ in Ω_b, $\mathscr{P}$ is elliptic in $\Omega_a \cup \Omega_b$. We can say:

$\mathscr{P}$ is interior elliptic in $\Omega \setminus \Gamma$.

In order to prove Th. 2.1, we explain how the condition

" a, b, and $a+b$ invertible on Γ "

can be viewed as an interface ellipticity.

Interface ellipticity needs 3 symbols associated to each x_0 in the interface Γ

- For $x_0 \in \Gamma$, we define the two limit interior symbols of $\mathscr{P}$ by (cf slide 11)

$$\sigma_{\mathrm{I},a}(x_0,\xi) = a(x_0)|\xi|^2 \qquad \text{and} \qquad \sigma_{\mathrm{I},b}(x_0,\xi) = b(x_0)|\xi|^2$$

  a and b invertible means that the interior ellipticity holds up to the interface, on both sides.
- For $x_0 \in \Gamma$, we need a new operator-valued symbol $\sigma_{\mathrm{T}}(x_0)$ that takes into account the transmission condition.

Transmission symbols

Take Ω as an open set in $\mathbb{R}^d$, $d = 2, 3, \ldots$, split into two subdomains Ω_a and Ω_b by a smooth interface $\Gamma \subset \Omega$.

For x_0 chosen in Γ define:

- Local (tangential, normal) coordinates $(x', t) \in \mathbb{R}^{d-1} \times \mathbb{R}$ around x_0 so that
    - $t < 0$ in Ω_b,
    - $t = 0$ in Γ,
    - $t > 0$ in Ω_a,
    - the local map $\psi : x \to (x', t)$ satisfies $D\psi(x_0) = \mathbb{I}$.
- The dual variable of x' as $\xi' \in \mathbb{R}^{d-1}$.

For $\xi' \in \mathbb{R}^{d-1}$, the transmission symbol $\sigma_{\mathrm{T}}(x_0, \xi')$ is defined in variational form as:

$$\begin{array}{rccc} \sigma_{\mathrm{T}}(x_0,\xi') : & H^1(\mathbb{R}) & \longrightarrow & H^{-1}(\mathbb{R}) \\ & u & \longmapsto & \left(v \ni H^1(\mathbb{R}) \mapsto \mathscr{A}_{\mathrm{T}}[x_0,\xi'](u,v)\right) \end{array}$$

with the form $\mathscr{A}_{\mathrm{T}}[x_0, \xi']$ obtained by partial Fourier transform $x' \mapsto \xi'$:

$$\mathscr{A}_{\mathrm{T}}[x_0,\xi'](u,v) = b(x_0)\int_{-\infty}^{0}(\partial_t u\,\partial_t v + |\xi'|^2 uv)\,\mathrm{d}t + a(x_0)\int_{0}^{+\infty}(\partial_t u\,\partial_t v + |\xi'|^2 uv)\,\mathrm{d}t$$

Interface ellipticity

$\Omega \subset \mathbb{R}^d$ with a smooth interface $\Gamma \subset \Omega$.

Definition

We say that $\mathscr{P}$ is interface elliptic on Γ if (compare with slide 11)

$(*)$ $\quad \forall x \in \Gamma, \quad \forall \xi \in \mathbb{S}^{d-1}, \quad \sigma_{\mathrm{I},a}(x,\xi),\ \sigma_{\mathrm{I},b}(x,\xi)$ are invertible,

and

$(**)$ $\quad \forall x \in \Gamma, \quad \forall \xi' \in \mathbb{S}^{d-2}, \quad \sigma_{\mathrm{T}}(x,\xi')$ is invertible.

Lemma

Assume $(*)$, i.e. a and b are invertible on Γ. Let $x \in \Gamma$ and $\xi' \in \mathbb{S}^{d-2}$.
Then $\sigma_{\mathrm{T}}(x,\xi')$ is Fredholm of index 0. Here is its kernel:

1. if $a(x) + b(x) \neq 0$, $\ker \sigma_{\mathrm{T}}(x,\xi') = \{0\}$
2. if $a(x) + b(x) = 0$, $\ker \sigma_{\mathrm{T}}(x,\xi')$ is generated by function $t \mapsto w[x,\xi'](t)$

$$w[x,\xi'](t) = e^{|\xi'|t}, \quad t \le 0 \quad \text{and} \quad w[x,\xi'](t) = e^{-|\xi'|t}, \quad t \ge 0$$

Proof of Lemma: The kernel

The Fredholm property is a consequence of ellipticity $(*)$ combined with a compact perturbation argument. Symmetry gives the zero index.

Denote by u_a, u_b the restrictions to $(0, \infty)$ and $(-\infty, 0)$ of a function u. An element $u = (u_a, u_b)$ of $\ker \sigma_{\mathrm{T}}(x, \xi')$ satisfies

$$\begin{cases} a(x)\,(-\partial_t^2 + |\xi'|^2)u_a = 0 & \text{in} \quad (0, +\infty) \\ b(x)\,(-\partial_t^2 + |\xi'|^2)u_b = 0 & \text{in} \quad (-\infty, 0) \\ -a(x)\,\partial_t u_a + b(x)\,\partial_t u_b = 0 & \text{at} \quad t = 0 \end{cases}$$

We find

$$u_a(t) = \alpha e^{-|\xi'|t} + \alpha' e^{|\xi'|t} \quad \text{and} \quad u_b(t) = \beta e^{|\xi'|t} + \beta' e^{-|\xi'|t}$$

Since $u \in H^1(\mathbb{R})$:

- u_a and u_b cannot be exponentially increasing, hence $\alpha' = \beta' = 0$.
- u_a and u_b have to coincide at $t = 0$, hence $\alpha = \beta$.

The last equation gives

$$(a(x) + b(x))\alpha|\xi'| = 0$$

This proves the lemma. □

Elliptic $\Longrightarrow$ Fredholm

Theorem 2.2

Under hyp. of slide 10, assume moreover that the interface Γ is $\mathscr{C}^\infty$. Then

$$\mathscr{P} \text{ Fredholm } H^1(\Omega) \to H^1(\Omega)' \quad \Longleftrightarrow \quad \mathscr{P} \text{ interface elliptic on } \Gamma.$$

Proof of "Elliptic $\Longrightarrow$ Fredholm"

By construction of a parametrix $\mathscr{E}$. As in the interior case, the essential ingredient is the existence, for any $x_0 \in \overline{\Omega}$, of an invertible operator $\widetilde{\mathscr{L}}_{x_0}$ that has the same principal part at x_0 as $\mathscr{P}$.

- If $x_0 \in \Omega_a \cup \Omega_b$, this is the interior case, slide 12.
- If $x_0 \in \partial\Omega$, this is the standard Neumann bc case.
- If $x_0 \in \Gamma$, we set

$$\widetilde{\mathscr{L}}_{x_0} := \sigma_{\mathrm{T}}(x_0, -i\partial_{y'} + \tfrac{1}{2})$$

Then, the invertibility of $\sigma_{\mathrm{T}}(x_0, \xi')$ for all nonzero $\xi' \in \mathbb{R}^{d-1}$ yields that

$$\widetilde{\mathscr{L}}_{x_0} \quad \text{invertible} \quad H^1(\mathbb{T}^{d-1} \times \mathbb{R}) \to H^{-1}(\mathbb{T}^{d-1} \times \mathbb{R})$$

Non-Elliptic $\Longrightarrow$ Non-Fredholm

Proof of "Non-Elliptic $\Longrightarrow$ Non-Fredholm"

By construction of Weyl sequences (aka quasimodes).
Pick $\chi \in \mathscr{C}_0^\infty(-\frac{1}{2}, \frac{1}{2})$ such that $\chi \equiv 1$ on $[-\frac{1}{4}, \frac{1}{4}]$.
There exists x_0 such that either

- $\sigma_{\mathrm{I},a}(x_0, \xi)$ or $\sigma_{\mathrm{I},b}(x_0, \xi)$ is not invertible. Say $\sigma_{\mathrm{I},a}(x_0, \xi)$. This means that $a(x_0) = 0$. Take $\xi_0 = (\xi_0', \tau) \in \mathbb{S}^{d-1}$ and construct "sliding quasimodes"

$$u_n(x', t) = \chi\left(\sqrt{n}\,|x' - x_0'|\right)\chi(\sqrt{n}\,t - 1)\, e^{in\langle \xi_0', x'\rangle}\, e^{in\tau t}.$$

- there exists $\xi_0' \in \mathbb{S}^{d-2}$ such that $\sigma_{\mathrm{T}}(x_0, \xi_0')$ is not invertible. This means that $a(x_0) + b(x_0) = 0$. Hence the existence of nonzero kernel elements $t \mapsto w[x_0, \xi'](t)$. Take $\xi_0' \in \mathbb{S}^{d-2}$ and construct "sitting quasimodes"

$$u_n(x', t) = \chi\left(\sqrt{n}\,|x' - x_0'|\right)\chi(\sqrt{n}|t|)\, e^{in\langle \xi_0', x'\rangle}\, w[x_0, \xi_0'](t).$$

Notions of sliding and sitting quasimodes borrowed from

V. Bonnaillie-Noël, M. Dauge, N. Popoff
Ground state energy of the magnetic Laplacian on corner domains.
Mém. Soc. Math. Fr., No. 145. 2016.

Conclusion for interface ellipticity

Then Theorem 2.2 is proved.

Combined with the Lemma slide 17, this proves Theorem 2.1.

Remark

The interface symbol $\sigma_{\mathrm{T}}(x,\xi')$ can be defined in an alternative form.
Set $PH^2(\mathbb{R})$ the subspace of $L^2(\mathbb{R})$ functions u such that $u_a \in H^2(\mathbb{R}_+)$ and $u_b \in H^2(\mathbb{R}_-)$, with u_a and u_b the restrictions of u to $\mathbb{R}_+$ and $\mathbb{R}_-$

$$\sigma^{\text{alt}}_{\mathrm{T}}(x,\xi') : \quad \begin{array}{ccc} H^1(\mathbb{R}) \cap PH^2(\mathbb{R}) & \longrightarrow & L^2(\mathbb{R}_+) \times L^2(\mathbb{R}_-) \times \mathbb{R} \\ u & \longmapsto & (f_a, f_b, g) \end{array}$$

with

$$\begin{cases} a(x)\,(-\partial_t^2 + |\xi'|^2)u_a = f_a & \text{in} \quad \mathbb{R}_+ \\ b(x)\,(-\partial_t^2 + |\xi'|^2)u_b = f_b & \text{in} \quad \mathbb{R}_- \\ -a(x)\,\partial_t u_a + b(x)\,\partial_t u_b = g & \text{at} \quad t=0 \end{cases}$$

We have the equivalence for each $x \in \Gamma$ and $\xi' \in \mathbb{R}^{d-1}$

$$\sigma_{\mathrm{T}}(x,\xi') \text{ invertible} \iff \sigma^{\text{alt}}_{\mathrm{T}}(x,\xi') \text{ invertible}$$

Moreover $\mathscr{P}$ is Fredholm from $H^1(\Omega) \to H^1(\Omega)'$ if and only if $\mathscr{P}$ is Fredholm from $H^1(\Omega) \cap PH^2(\Omega) \to L^2(\Omega_a) \times L^2(\Omega_b) \times H^{\frac{1}{2}}(\Gamma)$.

Outline for current section

Corner ellipticity

Under hypotheses of slide 10, assume that the interface Γ is curvilinear polygonal: This means that at each point $x_0 \in \Gamma = \partial\Omega_b$, the domain Ω_b is locally smoothly diffeomorphic to a neighborhood of 0 in either

1. a half-plane $\mathbb{R} \times \mathbb{R}_+$ (x_0 is a smooth transmission point)
2. a plane sector $K_b[x_0]$ (x_0 is a corner – aka vertex)

Denote by $\mathfrak{C}$ the set of corners. $\mathfrak{C}$ is finite. We introduce the definition of corner ellipticity.

Definition

We say that $\mathscr{P}$ is corner elliptic on $\mathfrak{C}$ if (compare with slide 17)

$$(*) \qquad \forall x \in \mathfrak{C}, \quad \forall \xi \in \mathbb{S}^1, \quad \sigma_{\mathrm{I},a}(x,\xi),\ \sigma_{\mathrm{I},b}(x,\xi) \text{ are invertible},$$

$$(**) \qquad \forall x \in \mathfrak{C}, \quad \forall \xi' \in \mathbb{S}^0, \quad \sigma_{\mathrm{T},\pm}(x,\xi') \text{ is invertible},$$

$$(***) \qquad \forall x \in \mathfrak{C}, \quad \sigma_{\mathrm{C}}(x) \text{ is invertible}.$$

- $\sigma_{\mathrm{T},\pm}$ are the two interface symbols on the two sides of the sector
- $\sigma_{\mathrm{C}}(x)$ defined on next slide.

Corner symbol

Choose x_0 in the corner set $\mathfrak{C}$.
Define local coordinates t around x_0, with $t = (t_1, t_2)$, so that:

- $t = 0$ represents the corner x_0
- $t \in K_b[x_0]$ represents points in Ω_b
- $t \in K_a[x_0]$ represents points in Ω_a, where $K_a[x_0] = \mathbb{R}^2 \setminus \overline{K_b}[x_0]$
- The local map $\psi : x \to t$ satisfies $D\psi(x_0) = \mathbb{I}$.

For x_0 chosen in $\mathfrak{C}$, the symbol $\sigma_{\mathrm{C}}(x_0)$ is the frozen operator $-\operatorname{div} A(x_0)\,\mathbf{grad}$ in variational form acting in the unit disk $\mathbb{D}^2$:

$$\begin{array}{rccc} \sigma_{\mathrm{C}}(x_0): & H_0^1(\mathbb{D}^2) & \longrightarrow & H^{-1}(\mathbb{D}^2) \\ & u & \longmapsto & \left(v \ni H^1(\mathbb{D}^2) \mapsto \mathscr{A}_{\mathrm{C}}[x_0](u,v)\right) \end{array}$$

with

$$\mathscr{A}_{\mathrm{C}}[x_0](u,v) = a(x_0) \int_{\mathbb{D}_a[x_0]} \nabla_t u \cdot \nabla_t v \,\mathrm{d}t + b(x_0) \int_{\mathbb{D}_b[x_0]} \nabla_t u \cdot \nabla_t v \,\mathrm{d}t$$

with the finite sectors

$$\mathbb{D}_a[x_0] = K_a[x_0] \cap \mathbb{D}^2 \quad \text{and} \quad \mathbb{D}_b[x_0] = K_b[x_0] \cap \mathbb{D}^2$$

Global ellipticity

Definition of (global) ellipticity

Assume that Γ is curvilinear polygonal. We say that $\mathscr{P}$ is elliptic if it is

1. interior elliptic in $\Omega \setminus \Gamma$
2. interface elliptic on $\Gamma \setminus \mathfrak{C}$
3. corner elliptic on $\mathfrak{C}$

We already know that

1. $a > 0$ in Ω_a & $b < 0$ in Ω_b $\Longrightarrow$
   $\mathscr{P}$ interior elliptic in $\Omega \setminus \Gamma$
2. $a > 0$ in $\overline{\Omega}_a$, $b < 0$ in $\overline{\Omega}_b$, and $a + b$ invertible on $\Gamma \setminus \mathfrak{C}$ $\Longrightarrow$
   $\mathscr{P}$ interface elliptic on $\Gamma \setminus \mathfrak{C}$
3. $a > 0$ in $\overline{\Omega}_a$, $b < 0$ in $\overline{\Omega}_b$, and $a + b$ invertible on Γ $\Longrightarrow$
   $(*)$ and $(**)$ of corner ellipticity on $\mathfrak{C}$

Criterion for invertibility of corner symbol

For $x_0 \in \mathfrak{C}$, let $\omega_b(x_0)$ be the opening of sector $K_b[x_0]$, well defined due to:

$$\text{local map } \psi : x \to t \text{ satisfies } D\psi(x_0) = \mathbb{I}$$

Then the opening of $K_a[x_0]$ is $\omega_a[x_0] = 2\pi - \omega_b[x_0]$. Set

$$\omega[x_0] = \max\{\omega_a[x_0], \omega_b[x_0]\}$$

Theorem 3.1

Let $x_0 \in \mathfrak{C}$. The corner symbol $\sigma_{\mathrm{C}}(x_0)$ is invertible if and only if

$$(\blacklozenge) \qquad \frac{a(x_0)}{b(x_0)} \notin \left[\frac{\omega[x_0]}{\omega[x_0] - 2\pi}, \frac{\omega[x_0] - 2\pi}{\omega[x_0]}\right]$$

Remarks

- If $b(x_0) \neq 0$, the condition $a(x_0) + b(x_0) \neq 0$ is equivalent to $\frac{a(x_0)}{b(x_0)} \neq -1$.
- We have $\frac{\omega[x_0]}{\omega[x_0]-2\pi} < -1$ and $\frac{\omega[x_0]-2\pi}{\omega[x_0]} > -1$
- Example of $\omega_b = \frac{\pi}{2}$: Then $\omega = \frac{3\pi}{2}$ and the forbidden interval is

$$\left[-3, -\tfrac{1}{3}\right]$$

($\blacklozenge$) $\Longrightarrow$ Invertibility of corner symbol

Based on the construction of an operator T inducing T-coercivity. A general form proposed in [BoChCi2012] is

$$Tu = \begin{cases} u_a & \text{in } \Omega_a \\ -u_b + 2R_a u_a & \text{in } \Omega_b \end{cases} \quad \text{or} \quad Tu = \begin{cases} u_a - 2R_b u_b & \text{in } \Omega_a \\ -u_b & \text{in } \Omega_b \end{cases}$$

with R_a an extension operator from $H^1(\Omega_a)$ to $H^1(\Omega)$, and similar for R_b.

Explicit formulas for R_a & R_b when Ω_a & Ω_b are finite sectors, e.g. for

$$\Omega_a = \mathbb{D}_a = \{x,\ r < 1,\ \theta \in (0, \alpha)\} \quad \text{and} \quad \Omega_b = \mathbb{D}_b = \{x,\ r < 1,\ \theta \in (\alpha, 2\pi)\}$$

are given in [BCC2012]: The extension R_a is defined as

$$R_a u_a(r,\theta) = \begin{cases} u_a(r,\theta) & \theta \in [0,\alpha] \\ u_a\left(r, \frac{\alpha}{2\pi-\alpha}(2\pi-\theta)\right), & \theta \in [\alpha, 2\pi] \end{cases}$$

Then

$$\|\nabla R_a u_a\|_{L^2(\mathbb{D}_b)} \leq \max\{\tfrac{\alpha}{2\pi-\alpha}, \tfrac{2\pi-\alpha}{\alpha}\} \|\nabla u_a\|_{L^2(\mathbb{D}_a)}$$

from which we deduce that if ($\blacklozenge$) holds then $\mathscr{A}_{\mathrm{C}}[x_0](u, Tu) \geq \beta |u|^2_{H^1(\Omega)}$.

Non-(♦) $\Longrightarrow$ Non-Invertibility of corner symbol: Mellin transform

- In the 2D situation, we have the corner symbols $\sigma_{\mathrm{C}}(x_0)$ (for each $x_0 \in \mathfrak{C}$) in the form of localized frozen operators.
- In the 3D case, in presence of edges, we meet actual partial-Fourier symbols including the dual variable ξ'' of edge abscissa, defining suitable edge symbols.
- In any case, to go further, we have to follow the path opened by [Kondrat'ev 1967] and introduce the *corner Mellin symbol* $\mathscr{M}_{\mathrm{C}}[x_0](\lambda)$ *of* $\sigma_{\mathrm{C}}(x_0)$. The distinctive feature of such symbol is that it is based on Mellin transf. instead of Fourier transf.

Definition of Mellin transform

For $x \in \mathbb{R}^d$, let $r = |x|$ and $\theta = \frac{x}{r} \in \mathbb{S}^{d-1}$. Let $u \in L^2(\mathbb{R}^d)$ with support $\operatorname{supp} u \Subset \mathbb{R}^d \setminus \{0\}$. Denote $\widetilde{u}$ its representation in polar coordinates r, θ.

The Mellin transform $\mathfrak{M}u : \mathbb{C} \ni \lambda \mapsto \mathfrak{M}[\lambda]u$ of u is defined for any $\lambda \in \mathbb{C}$ by

$$\mathfrak{M}[\lambda]u(\theta) = \int_0^{+\infty} r^{-\lambda}\widetilde{u}(r,\theta)\frac{\mathrm{d}r}{r}$$

The function $\lambda \mapsto \mathfrak{M}[\lambda]u$ is holomorphic on $\mathbb{C}$ with values in $L^2(\mathbb{S}^{d-1})$ and

$(\star)$ $\quad \mathfrak{M}[\lambda](\partial_\theta \widetilde{u}) = \partial_\theta\big(\mathfrak{M}[\lambda]u\big)$ and $\mathfrak{M}[\lambda](r\partial_r u) = \lambda\,\mathfrak{M}[\lambda]u$

Non-(♦) $\Longrightarrow$ Non-Invertibility of corner symbol: Mellin calculus

Pick $x_0 \in \mathfrak{C}$ and $u, v \in \mathscr{C}^\infty(\mathbb{D}^2 \setminus \{0\}$. Recall that the corner symbol $\sigma_{\mathrm{C}}(x_0)$ is defined via the bilinear form

$$\mathscr{A}_{\mathrm{C}}[x_0](u,v) \;=\; a(x_0)\int_{K_a[x_0]} \nabla_t u\cdot\nabla_t v\,\mathrm{d}t \;+\; b(x_0)\int_{K_b[x_0]} \nabla_t u\cdot\nabla_t v\,\mathrm{d}t$$

In polar coordinates

$$\begin{aligned}\mathscr{A}_{\mathrm{C}}[x_0](u,v) \;=\;& a(x_0)\int_0^{\omega_a[x_0]}\int_0^{+\infty}\frac{1}{r^2}\Big(\partial_\theta\widetilde{u}\,\partial_\theta\widetilde{v} + (r\partial_r\widetilde{u})\,(r\partial_r\widetilde{v})\Big)\,r\mathrm{d}r\mathrm{d}\theta\\ &+\; b(x_0)\int_{\omega_a[x_0]}^{2\pi}\int_0^{+\infty}\frac{1}{r^2}\Big(\partial_\theta\widetilde{u}\,\partial_\theta\widetilde{v} + (r\partial_r\widetilde{u})\,(r\partial_r\widetilde{v})\Big)\,r\mathrm{d}r\mathrm{d}\theta\end{aligned}$$

Relying on [integration by parts and] identities ($\star$) we find the bilinear form $\mathscr{N}_{\mathrm{C}}[x_0,\lambda]$ of the Mellin symbol $\mathscr{M}_{\mathrm{C}}[x_0](\lambda) : H^1(\mathbb{T}) \longrightarrow H^{-1}(\mathbb{T})$ of $\sigma_{\mathrm{C}}(x_0)$:

For $\varphi, \psi \in H^1(\mathbb{T})$,

$$\begin{aligned}\mathscr{N}_{\mathrm{C}}[x_0,\lambda](\varphi,\psi) = {}& a(x_0)\int_0^{\omega_a[x_0]}(\partial_\theta\varphi\,\partial_\theta\psi - \lambda^2\varphi\psi)\,\mathrm{d}\theta\\ &+ b(x_0)\int_{\omega_a[x_0]}^{2\pi}(\partial_\theta\varphi\,\partial_\theta\psi - \lambda^2\varphi\psi)\,\mathrm{d}\theta\end{aligned}$$

Non-(◆) $\Longrightarrow$ Non-Invertibility of corner symbol: Mellin symbol

In strong form the corner Mellin symbol $\mathscr{M}_{\mathrm{C}}[x_0](\lambda)$ is *(with $\omega_a[x_0]$ abbreviated into ω)*

$$\begin{array}{rccc} \mathscr{M}_{\mathrm{C}}(\lambda): & H^1(\mathbb{T}) \cap PH^2((0,\omega),(\omega,2\pi)) & \longrightarrow & L^2(0,\omega)\times L^2(\omega,2\pi)\times\mathbb{C}^2 \\ & \varphi & \longmapsto & (f_a, f_b, g_0, g_1) \end{array}$$

with

$$\begin{cases} a(x_0)\,(-\partial_\theta^2-\lambda^2)\varphi_a = f_a & \text{in} \quad (0,\omega) \\ b(x_0)\,(-\partial_\theta^2-\lambda^2)\varphi_b = f_b & \text{in} \quad (\omega,2\pi) \\ -a(x_0)\,\partial_\theta\varphi_a + b(x_0)\,\partial_\theta\varphi_b = g_0 & \text{at} \quad \theta = 0 \\ -a(x_0)\,\partial_\theta\varphi_a + b(x_0)\,\partial_\theta\varphi_b = g_1 & \text{at} \quad \theta = \omega \end{cases}$$

Ellipticity at the lower level (interior and interface, i.e. $(*)$ and $(**)$ of corner ellipticity) implies that the symbol $\lambda \mapsto \mathscr{M}_{\mathrm{C}}[x_0](\lambda)$ has a meromorphic inverse.

The poles of the inverse are the λ such that $\ker \mathscr{M}_{\mathrm{C}}[x_0](\lambda) \neq \{0\}$: Calculate

$$\varphi_a = \alpha e^{i\lambda\theta} + \alpha' e^{-i\lambda\theta} \quad \text{and} \quad \varphi_b = \beta e^{i\lambda\theta} + \beta' e^{-i\lambda\theta}$$

The 2 compatibility and 2 transmission conditions yield the 4×4 system for $(\alpha,\alpha',\beta,\beta')^\top$:

$$\begin{cases} \alpha + \alpha' = \beta e^{2i\pi\lambda} + \beta' e^{-2i\pi\lambda} \\ \alpha e^{i\omega\lambda} + \alpha' e^{-i\omega\lambda} = \beta e^{i\omega\lambda} + \beta' e^{-i\omega\lambda} \\ a(x_0)\,(\alpha - \alpha') = b(x_0)\,(\beta e^{2i\pi\lambda} - \beta' e^{-2i\pi\lambda}) \\ a(x_0)\,(\alpha e^{i\omega\lambda} - \alpha' e^{-i\omega\lambda}) = b(x_0)\,(\beta e^{i\omega\lambda} - \beta' e^{-i\omega\lambda}) \end{cases}$$

Non-(♦) $\Longrightarrow$ Non-Invertibility of corner symbol: Mellin spectrum

Set $\mu = \frac{a(x_0)}{b(x_0)}$. Then $\mathscr{M}_{\mathrm{C}}[x_0](\lambda)$ has a non-trivial kernel iff

$$\begin{vmatrix} 1 & 1 & -e^{2i\pi\lambda} & -e^{-2i\pi\lambda} \\ e^{i\omega\lambda} & e^{-i\omega\lambda} & -e^{i\omega\lambda} & -e^{-i\omega\lambda} \\ \mu & -\mu & -e^{2i\pi\lambda} & e^{-2i\pi\lambda} \\ \mu e^{i\omega\lambda} & -\mu e^{-i\omega\lambda} & -e^{i\omega\lambda} & e^{-i\omega\lambda} \end{vmatrix} = 0$$

i.e. iff

(♠) $$(\mu+1)\sin\lambda\pi = \pm(\mu-1)\sin\lambda(\pi-\omega)$$

M. Costabel, E. Stephan
A direct boundary integral equation method for transmission problems,
J. Math. Anal. Appl. 106 (1985) no. 2, 367-413.

A.-S. Bonnet-Ben Dhia, M. Dauge, K. Ramdani
Analyse spectrale et singularités d'un problème de transmission non coercif,
C. R. Acad. Sci. Paris Sér. I Math. 328 (1999) no. 8, 717-720.

When, $a(x_0)$, $b(x_0)$ and $a(x_0)+b(x_0)$ are nonzero, $\lambda \mapsto \mathscr{M}_{\mathrm{C}}[x_0](\lambda)$ has a meromorphic inverse, with poles located at the set $\Sigma_{\mathrm{C}}[x_0]$ of roots of eq. (♠).

$\Sigma_{\mathrm{C}}[x_0]$ is the spectrum of $\mathscr{M}_{\mathrm{C}}[x_0]$.

Non-(♦) $\Longrightarrow$ Non-Invert. of edge symbol: Threshold singularities

Recall —with $\omega = \max\{\omega_a, 2\pi - \omega_a\}$ and $\mu = \frac{a(x_0)}{b(x_0)}$,

(♠) $$(\mu + 1)\sin\lambda\pi = \pm(\mu - 1)\sin\lambda(\pi - \omega)$$

The roots of (♠) are either real or pure imaginary. Pure imaginary roots $\lambda = i\varkappa$ with $\varkappa \in \mathbb{R} \setminus \{0\}$ are present iff $\mu \in \left(\frac{\omega}{\omega - 2\pi}, \frac{\omega - 2\pi}{\omega}\right)$, i.e. when[1] (♦) is not true.

Let $\chi \in \mathscr{C}_0^\infty(\mathbb{D}^2)$ be a smooth cutoff function, $\chi \equiv 1$ in a neighborhood of 0.

Lemma

Let $\lambda = i\varkappa$ with $\varkappa \in \mathbb{R} \setminus \{0\}$ be a root of (♠). There exists nonzero $\theta \mapsto \varphi_\varkappa(\theta)$ in $\ker \mathscr{M}[x_0](\lambda)$, giving rise to the *"threshold singularity"*

$$U(r, \theta) = r^{i\varkappa}\varphi_\varkappa(\theta),$$

i.e. satisfying

- the function χU does not belong to $H_0^1(\mathbb{D}^2)$,
- $\sigma_{\mathrm{C}}(x_0)(\chi U) \in H^{-1}(\mathbb{D}^2)$,
- for any $n \in \mathbb{N}$, $r^{\frac{1}{n}}\chi U$ belongs to $H_0^1(\mathbb{D}^2)$.

Then the sequence $\left(r^{\frac{1}{n}}\chi U\right)_{n \geq 1}$ is a Weyl sequence for $\sigma_{\mathrm{C}}(x_0)$.

[1] When μ belongs to one of the boundaries of the forbidden interval, there is no pure imaginary root, but an anomalous pole in 0.

Conclusion for curvilinear polygonal interfaces

Recall that

$$\overline{\Omega} = \overline{\Omega}_a \cup \overline{\Omega}_b, \quad \Omega_a \cap \Omega_b = \emptyset \quad \text{and} \quad \Omega_b \Subset \Omega$$

and

$$\mathscr{P} = -\operatorname{div} A \,\mathbf{grad} \quad \text{with} \quad A = a > 0 \ \text{ in } \ \Omega_a \cup \partial\Omega, \ A = b < 0 \ \text{ in } \ \Omega_b$$

with smooth functions a and b.

We have (almost) proved:

Theorem 3.2

Assume that Γ is curvilinear polygonal. Then the following statements are equivalent:

1. $\mathscr{P}$ is Fredholm from $H^1(\Omega)$ into $H^1(\Omega)'$
2. $\mathscr{P}$ is elliptic in the sense of slide 24
3. The functions a and b satisfy
   - $\forall x_0 \in \Gamma$, $a(x_0)$, $b(x_0)$ and $a(x_0) + b(x_0)$ are nonzero.
   - $\forall x_0 \in \mathfrak{C}$, $\frac{a(x_0)}{b(x_0)} \notin \left[\frac{\omega[x_0]}{\omega[x_0]-2\pi}, \frac{\omega[x_0]-2\pi}{\omega[x_0]}\right]$

Outline for current section

Regularity shift

Recall

- Γ is a polygonal interface (2D case)
- $\mathfrak{C}$ is the set of corners x_0
- For each $x_0 \in \mathfrak{C}$, the spectrum of the corner symbol $\sigma_{\mathrm{C}}(x_0)$ is $\Sigma_{\mathrm{C}}[x_0]$

Theorem 4.1

Assume that $\mathscr{P}$ is interior and interface elliptic (up to corners), and

(♥) $\quad \forall x_0 \in \mathfrak{C}, \quad \Sigma_{\mathrm{C}}[x_0]$ is disjoint from the strip $\{\lambda \in \mathbb{C},\ \mathrm{Re}\,\lambda \in (0,1]\}$

Let $u \in H^1(\Omega)$ be such that $\mathscr{P}u \in L^2(\Omega)$. Then $u_a \in H^2(\Omega_a)$ and $u_b \in H^2(\Omega_b)$.

Remarks

- The condition $\mathscr{P}u = f \in L^2(\Omega)$ means

$$\begin{cases} -\operatorname{div} a\,\mathbf{grad}\,u = f & \text{in} \quad \Omega_a \\ -\operatorname{div} b\,\mathbf{grad}\,u = f & \text{in} \quad \Omega_b \\ a\,\partial_n u = 0 & \text{on} \quad \partial\Omega \\ a\,\partial_{n_a} u + b\,\partial_{n_b} u = 0 & \text{on} \quad \Gamma \end{cases}$$

- The elliptic regularity shift $L^2 \to PH^2$ is valid outside corners, i.e. in a neighborhood of any point $x_0 \in \overline{\Omega} \setminus \mathfrak{C}$.
- Condition (♥) holds e.g. if $\omega = \frac{\pi}{2}$ and $\mu \in (-3, -\frac{1}{3})$ (the forbidden interval for Fredholmness $H^1 \to H^{-1}$!)

Expansion

Theorem 4.2

Let $\mathscr{P}$ be interior and interface elliptic (up to corners). Assume, instead of (♥)

(♥$^\flat$) $\forall x_0 \in \mathfrak{C}, \quad \Sigma_{\mathbb{C}}[x_0]$ is disjoint from the line $\operatorname{Re}\lambda = 1$.

Let $u \in H^1(\Omega)$ be such that $\mathscr{P}u \in L^2(\Omega)$. Then u has an expansion around each corner x_0 according to

$$u = u_{\text{reg}} + \sum_{x_0 \in \mathfrak{C}} \sum_{\substack{\lambda \in \Sigma_{\mathbb{C}}[x_0] \\ 0 < \operatorname{Re}\lambda < 1}} \chi_{x_0} \, |x - x_0|^\lambda \varphi_{x_0,\lambda}$$

where

- the smooth cut-off χ_{x_0} localizes around x_0
- the term $\varphi_{x_0,\lambda}$ is a function of the angle $\theta_{x_0} = \frac{x-x_0}{|x-x_0|} \in \mathbb{T}$ and belongs to $\ker \mathscr{M}_{\mathbb{C}}[x_0](\lambda)$
- u_{reg} belongs to $PH^2(\Omega)$.

Remarks

- Theorem 4.2 implies Theorem 4.1.
- The cardinal of the set $\Sigma_{\mathbb{C}}[x_0] \cap \{\operatorname{Re}\lambda \in (0,1)\}$ is finite, due to ellipticity.
- $\ker \mathscr{M}_{\mathbb{C}}[x_0](\lambda)$ is finite dimensional, due to ellipticity.

Elements of proof: 0 – 2

0 Simplification.
For ease of exposition, we assume that in a neighborhood of each corner, coefficients a and b are constant and the sides of the interface Γ are straight.

1 Localization.
Using the local regularity away from the corners, we can use smooth cut-offs to isolate corners from each other. By mere translation and rotation, the localized at x_0 of the operator $\mathscr{P}$ coincides with its corner symbol $\sigma_{\mathrm{C}}(x_0)$:
Our assumption becomes

$$u \in H^1_0(\mathbb{D}^2), \quad \sigma_{\mathrm{C}}(x_0)(u) = f \in L^2(\mathbb{D}^2)$$

with $u \equiv 0$ in a neighborhood of $\partial\mathbb{D}^2$, so that we have

$$u \in H^1(\mathbb{R}^2),\ \operatorname{supp} u \Subset \mathbb{D}^2, \quad \sigma_{\mathrm{C}}(x_0)(u) = f \in L^2(\mathbb{R}^2)$$

2 Polar coordinates. In polar coordinates and strong form

$$\begin{cases} -a\big(\partial_\theta^2 + (r\partial_r)^2\big)\widetilde{u}_a = r^2\widetilde{f} & \text{in} \quad K_a \\ -b\big(\partial_\theta^2 + (r\partial_r)^2\big)\widetilde{u}_b = r^2\widetilde{f} & \text{in} \quad K_b \\ -a\,\partial_\theta\widetilde{u} + b\,\partial_\theta\widetilde{u} = 0 & \text{if} \quad \theta = 0 \ \text{ or } \ \theta = \omega \end{cases}$$

Elements of proof: 3 – 4

3 Mellin transform.
As u and f are in L^2 and have compact support their Mellin transforms are well defined for $\operatorname{Re}\lambda \leq -1$. Set

$$U(\lambda) = \mathfrak{M}[\lambda](u) \quad \text{and} \quad G(\lambda) = \mathcal{M}[\lambda](r^2 f)$$

Then $\sigma_{\mathrm{C}}(u) = f$ becomes

$$\mathcal{M}_{\mathrm{C}}(\lambda)\, U(\lambda) = G(\lambda), \quad \operatorname{Re}\lambda \leq -1.$$

In fact, due to relations between Mellin and Fourier-Laplace transforms via the change of variable $\mathbb{R}_+ \ni r \to \tau = \log r \in \mathbb{R}$:

- $G(\lambda)$ is defined for $\lambda \leq 1$ and holomorphic for $\operatorname{Re}\lambda < 1$ with values in $L^2(\mathbb{T})$
- $U(\lambda)$ is defined and holomorphic for $\lambda < 0$ with values in $H^1(\mathbb{T})$ (Hardy's inequality)

4 Meromorphic extension.
We define a meromorphic extension $U^\sharp$ of U to the strip $\operatorname{Re}\lambda \in [0, 1]$ by

$$U^\sharp(\lambda) = \mathcal{M}_{\mathrm{C}}(\lambda)^{-1} G(\lambda).$$

We have

- $U^\sharp$ is meromorphic with values in $PH^2((0, \omega), (\omega, 2\pi))$
- Possible poles of $U^\sharp$ belong to the spectrum Σ_{C} of $\mathcal{M}_{\mathrm{C}}$ and to the strip $\operatorname{Re}\lambda \in [0, 1]$.

Elements of proof: 5

5 Inverse Mellin transform.

The function $\mathbb{R} \ni \varkappa \mapsto G(\eta + i\varkappa) \in L^2(\mathbb{T})$ belongs to $L^2(\mathbb{R})$ for all $\eta \leq 1$.

On line $\operatorname{Re}\lambda = 1$ (disjoint from the spectrum Σ_{C}), the resolvent $\mathscr{M}_{\mathrm{C}}(\lambda)^{-1}$ satisfies weighted estimates (due to interior and interface ellipticity), that imply

$$(\mathscr{R}) \qquad |\lambda|^2 \, \|U^\sharp(\lambda)\|_{L^2(\mathbb{T})} + |\lambda| \, \|U^\sharp(\lambda)\|_{H^1(\mathbb{T})} + \|U^\sharp(\lambda)\|_{PH^2(\mathbb{T})} \leq C\|G(\lambda)\|_{L^2(\mathbb{T})}$$

with a constant C independent of λ, for $\operatorname{Re}\lambda = 1$.

The inverse Mellin formula

$$u_1(x) = \frac{1}{2i\pi}\int_{\operatorname{Re}\lambda=1} r^\lambda U^\sharp(\lambda)(\theta)\,\mathrm{d}\lambda$$

defines a function u_1 that satisfies

$$r^{-2}u_1 \in L^2(\mathbb{R}^2), \quad r^{-1}\nabla u_1 \in L^2(\mathbb{R}^2)$$

and, for its second order derivatives

$$\partial^\alpha u_1\big|_{K_a} \in L^2(K_a), \quad \text{and} \quad \partial^\alpha u_1\big|_{K_b} \in L^2(K_b), \quad |\alpha| = 2.$$

Elements of proof: 6

6 Residue formula.

We have

$$u_1 = \frac{1}{2i\pi} \int_{\mathrm{Re}\ \lambda=1} r^\lambda U^\sharp(\lambda)\, \mathrm{d}\lambda$$

and, for any $\varepsilon > 0$

$$u = \frac{1}{2i\pi} \int_{\mathrm{Re}\ \lambda=-\varepsilon} r^\lambda U^\sharp(\lambda)\, \mathrm{d}\lambda$$

The difference is given by the residue formula

$$u_1 - u = \frac{1}{2i\pi} \int_\gamma r^\lambda U^\sharp(\lambda)\, \mathrm{d}\lambda$$

where γ is any simple contour surrounding the poles of $U^\sharp(\lambda)$ within the strip $\mathrm{Re}\ \lambda \in [0, 1]$ (uses resolvent estimates $(\mathscr{R})$). Thus

$$u_1 - u = \sum_{\substack{\lambda_0 \in \Sigma_C \\ 0 \leq \mathrm{Re}\ \lambda_0 < 1}} \operatorname*{Res}_{\lambda=\lambda_0} r^\lambda U^\sharp(\lambda)$$

Elements of proof: End of proof

In order to finally prove our Theorem 4.2, it remains to

- Set $u_{\text{reg}} = \sum_{x_0 \in \mathfrak{C}} \chi_{x_0} u_{1,x_0}$
- Notice that the residues $\text{Res}_{\lambda=\lambda_0} r^\lambda U^\sharp(\lambda)$ correspond to the poles of $\mathcal{M}_{\mathrm{C}}(\lambda)^{-1}$. For $\lambda_0 \neq 0$, these poles are simple, thus are given by a projection $\Pi[\lambda_0]$ on the kernel of $\mathcal{M}_{\mathrm{C}}(\lambda_0)$:

$$\begin{aligned}\operatorname*{Res}_{\lambda=\lambda_0} r^\lambda U^\sharp(\lambda) &= \operatorname*{Res}_{\lambda=\lambda_0} r^\lambda \mathcal{M}_{\mathrm{C}}(\lambda)^{-1} G(\lambda) \\ &= r^{\lambda_0} \left(\operatorname*{Res}_{\lambda=\lambda_0} \mathcal{M}_{\mathrm{C}}(\lambda)^{-1} \right) G(\lambda_0) \\ &= r^{\lambda_0} \, \Pi[x_0] \, G(\lambda_0)\end{aligned}$$

- For $\lambda_0 = 0$, since we already know that $u \in H^1$, the only possible contribution is a constant.
- We deduce formula ($\heartsuit^\flat$).

More details in https://hal.archives-ouvertes.fr/cel-01399350 especially in sect. 5 (for Dirichlet boundary condition at a corner).

Outline for current section

Related spectral problems

1. Choose a and b such that $\mathscr{P}$ is elliptic, and look for the [semi-classical] discrete spectrum

$$-[h^2]\,\text{div}\,A\,\mathbf{grad}\,u = \Lambda u, \quad [h \to 0]$$

The numerical approximation of this spectrum is addressed in

C. Carvalho, L. Chesnel, P. Ciarlet Jr
Eigenvalue problems with sign-changing coefficients.
C. R. Math. Acad. Sci. Paris **355** (2017) no. 6, 671-675.

A.-S. Bonnet-Ben Dhia, C. Carvalho, P. Ciarlet Jr
Mesh requirements [...] with sign-changing coefficients.
Numer. Math. **138** (2018) no. 4, 801-838.

2. Set $a \equiv 1$ and consider $b =: \Lambda$ as a spectral parameter: We can define the *"opertor pencil"* $\mathfrak{P} : \mathbb{C} \ni \Lambda \mapsto \mathfrak{P}(\Lambda)$ as

$$\mathfrak{P}(\Lambda) = -\,\text{div}\,\mathbb{1}_{\Omega_a}\,\mathbf{grad}\,u - \Lambda\,\text{div}\,\mathbb{1}_{\Omega_b}\,\mathbf{grad}\,u \;:\; H^1(\Omega)/\mathbb{C} \to (H^1(\Omega)/\mathbb{C})'$$

with $\mathbb{1}$ denoting the characteristic function. The essential spectrum is the set of Λ's for which $\mathfrak{P}(\Lambda)$ is not Fredholm, *c.f.* Th.2.1 and 3.2. When $\Lambda \notin \mathbb{R}_-$, $\mathfrak{P}(\Lambda)$ is invertible. We expect that the rest of the spectrum is formed by eigenvalues, as in the smooth, unbounded case considered in

D. Grieser
The plasmonic eigenvalue problem.
Rev. Math. Phys. **26** (2014) no. 3.

Final conclusion

S Strength

Laplace-based example: The resolvent is more explicit.

Dimension 2: The T-coercivity is optimal.

W Weakness

Lack of coercivity. Unbounded operators in both directions.

O Opportunities

Possible generalization to 3D configurations, with a polyhedral interface.

T Threats

Too many possible generalizations...

Too many possible directions of investigation...